Lecture 5: Differentiation

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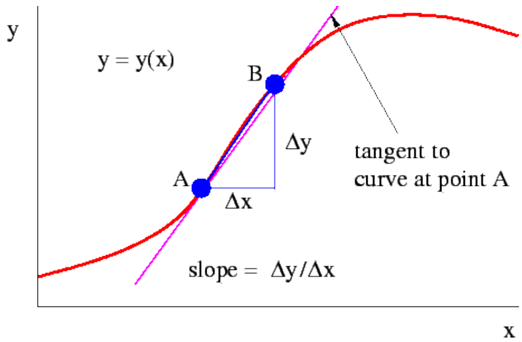
Slope of the tangent line

One of the main problems we seek to solve in calculus is the slope of the tangent at a specific point on f(x).
The slope of the tangent line at *f(a)* is the instantaneous rate of change of *f(x)* at that moment.



formula for the slope of the tangent:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(if the limit exists)

Example:
$f(x) = \frac{1}{x^2}$, slope at a=1

Use $a = 1$

$$m = \lim_{x \to 1} \frac{\frac{1}{x^2} - \frac{1}{a^2}}{x - a} \rightarrow = \lim_{x \to 1} \frac{\frac{1}{x^2} - \frac{1}{1^2}}{x - 1} \longrightarrow = \lim_{x \to 1} \frac{\frac{1}{x^2} - 1}{x - 1} \longrightarrow = \lim_{x \to 1} \left(\frac{\frac{1 - x^2}{x^2}}{x - 1} \right)$$

$$= \lim_{x \to 1} \frac{1 - x^2}{x^2(x - 1)} \rightarrow = \lim_{x \to 1} \frac{-\cancel{(1 - x)}(1 + x)}{x^2 \cdot \cancel{(x - 1)}} \rightarrow = \lim_{x \to 1} \frac{-(1 + x)}{x^2} \rightarrow = \frac{-(1 + 1)}{1^2} = 2$$

Velocity

Eg. for the following equation:
$s = 1.1t + 1.8t^2$
s is the distance in metres and *t* is the time elapsed of a book thrown in the air.
To calculate the:

final initial

$$average\ speed = \frac{\Delta s}{\Delta t} = \frac{s(f) - s(i)}{t(f) - t(i)}$$

and instantaneous velocity:

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This is also the derivative of a function, *f'* ("*f* prime"):

$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ --> derivative of *f* at *a* if limit exists

Example: $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{a^2}}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(2a + h)}{\cancel{h}}$$
$$= \lim_{h \to 0} 2a + h$$
$$= 2a$$

Equation of a tangent line:

$= 2a$

Equation of a tangent line:

Ex: $f(x) = \frac{1}{x^2}$, tangent line at x=3 ← a

We need to find the slope, which is equivalent to f'(3):

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(3+h)^2} - \frac{1}{3^2}}{h}$$

Find common denominator in numerator:

$$= \lim_{h\to 0} \frac{\frac{3^2 - (3+h)^2}{9 * (3+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{9 - (3^2 + 6h + h^2)}{9 \cdot (3+h)^2 \cdot h}$$

$$= \lim_{h\to 0} \frac{(-6h + h^2)}{9 \cdot (3+h)^2 \cdot h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}(6+h)}{9 \cdot (3+h)^2 \cdot \cancel{h}}$$

Substitute zero for *h*:

$$= \lim_{h\to 0} \frac{-6}{9 \cdot (3)^2}$$

$$= -\frac{6}{81}$$

$$= -\frac{2}{27}$$

Therefore, the slope of the tangent at x=3 is -2/27. To determine the equation of the tangent we need a point on the line:

$f(3) = \frac{1}{3^2}$, therefore $\left(\overset{a}{3}, \overset{f(a)}{\frac{1}{9}}\right)$ is a point on the line.

Equation of the tangent line:

$$y - f(a) = f'(a)(x - a)$$

$$y - \frac{1}{9} = -\frac{2}{27}(x - 3)$$

$$y = -\frac{2}{27}x - \frac{2}{9} + \frac{1}{9}$$

$$y = -\frac{2}{27}x - \frac{1}{3}$$

Derivative as a function; let a vary, use x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ if limit exists

OR

$$f'(x) = \frac{dy}{dx} = \frac{d}{dz} f(x) = Df(x)$$

A function is **differentiable at a** if f'(a) exists. It is **differentiable on an interval (a,b)** if it is differentiable at every point in (a,b).

<u>Theorem</u> If *f* is differentiable at a, then *f* is continuous at *a*.